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A STUDY ON THE FIRST ORDER DIFFERENTIAL ALGEBRAIC EQUATIONS USING THE RATIONAL APPROXIMATION METHOD

R. Pushpalatha Assistant Professor, Department of Mathematics, Jayagovind Harigopal Agarwal Agarsen College, Chennai – 600 060, Tamil Nadu, India. rplatha1908@gmail.com
S. Sekar Associate Professor, Department of Mathematics, Government Arts and Science College, Thittamalai – 638 458, Tamil Nadu, India. angelsekar@gmail.com

Abstract —
In this paper a study on the first order differential algebraic equations of time-invariant and time varying cases [6] using the rational approximation method is considered. The obtained discrete solutions using the rational approximation method are compared with the exact solutions of the first order differential algebraic equations of time-invariant and time varying cases and pade approximation method. Tables and graphs are presented to show the efficiency of this method. This rational approximation method can be easily implemented in a digital computer and the solution can be obtained for any length of time.

Keywords — First order differential algebraic equations of time-invariant and time-varying cases, Singular systems, Pade Approximation, Rational Approximation Method.

INTRODUCTION

Singular systems are being applied to solve a variety of problems involved in various disciplines of science and engineering. They are applied to analyse neurological events and catastrophic behaviour and they also provide a convenient form for the dynamical equations of large-scale interconnected systems. Further, singular systems are found in many areas such as constrained mechanical systems, fluid dynamics, chemical reaction kinetics, simulation of electrical networks, electrical circuit theory, power systems, aerospace engineering, robotics, aircraft dynamics, neural networks, neural delay systems, network analysis, time series analysis, system modelling, social systems, economic systems, biological systems etc.[5, 7-13]

Wazwaz [14] published a paper on modified Runge-Kutta formula based on a variety of means of third order. Murugesan et al. [1 - 4] have analysed different second-order systems and multivariable linear systems via RK method based on centroidal mean, and also, they extended RK formulae based on variety of means to solve system of IVPs. In this paper, we apply the rational approximation method for finding the numerical solution of first order differential algebraic equations of time-invariant and time varying cases with more accuracy.

RATIONAL APPROXIMATION METHOD

The basic principle in designing numerical methods for solving the initial value problem

$$y^{\cdot} = f(t, y), \quad y(t_0) = y_0 \qquad (1)$$

is that the numerical method must fit the Taylor series expansion of the solution in a given point with the desired accuracy. For a given initial value problem (1), where y is a vector-valued phase variable and t is a scalar (time), the approximate solution obtained in the next time point is denoted as $y_{n+1} \approx y(t_{n+1})$. Denote the time step $h = t_{n+1} - t_n$. Then, consider the Taylor series that approximates the solution:

$$y_{n+1} = \sum_{i=0}^{\infty} \frac{y_n^{(i)}}{i!} h^i. \tag{2}$$

Consider a rational approximation of y_{n+1} as a fraction of two polynomials $P_L(h)$ and $Q_M(h)$:

$$P_L(h) = p_0 + p_1 h + ... + p_L h^L,$$
$$Q_M(h) = q_0 + q_1 h + ... + q_M h^M,$$

where L and M are powers of these polynomials. The rational approximation of the solution reads:

$$y_{n+1} = \frac{P_L(h)}{Q_M(h)}. \tag{3}$$

A number of reliable algorithms for finding the rational approximant (3) exist [19]. Here, we use the most straightforward approach. Let M + L = p, where p is a certain natural number. Assuming that (2) and (3) must give the same result up to the error term $O(h^{p+1})$, it follows that

$$\sum_{i=0}^{\infty} \frac{y^{(i)}}{i!} h^i - \frac{P_L(h)}{Q_M(h)} = O(h^{p+1}), \tag{4}$$

which results in a system of equations:

$$\begin{cases} p_0 = y_n q_0, \\ p_1 = h y'_n q_0 + y_n q_1, \\ p_2 = h^2 \frac{y''_n}{2} q_0 + h y'_n q_1 + y_n q_2, \\ \vdots \\ p_L = h^L \frac{y_n^{(L)}}{L!} q_0 + h^{L-1} \frac{y_n^{(L-1)}}{(L-1)!} q_1 + \cdots + y_n q_L, \\ 0 = h^{L+1} \frac{y_n^{(L+1)}}{(L+1)!} q_0 + h^L \frac{y_n^{(L)}}{L!} q_1 + \cdots + h^{L-M+1} \frac{y_n^{(L-M+1)}}{(L-M+1)!} q_M, \\ \vdots \\ 0 = h^{L+M} \frac{y_n^{(L+M)}}{(L+M)!} q_0 + h^{L+M-1} \frac{y_n^{(L+M-1)}}{(L+M-1)!} q_1 + \cdots + h^L \frac{y_n^{(L)}}{L!} q_M. \end{cases} \tag{5}$$

One can notice that this system is complete only if L + M = p, which is exactly the case of the Padé rational approximation. Similarly, the Padé approximant corresponds to the rational approximation of the highest possible order of accuracy. Usually, the notation $R_{L/M}$ or [L/M] is used to denote the powers of polynomials in the Padé approximant with the power of the numerator L and the power of the denominator M. Otherwise, if L + M − p = r, then r > 0, the last r equations from (5) should be omitted, and the r coefficients in P and Q remain free. This option can be used to obtain rational approximants with some desired properties, but their order of accuracy is higher than that with the Padé approximant.

First Order Differential Algebraic Equations

In general a first order differential algebraic equations of time-invariant case is represented in the following form

$$K\dot{x}(t) = Ax(t) + Bu(t)$$

with initial condition $x(0) = x_0$.

where K is an $n \times n$ singular matrix, A and B are $n \times n$ and $n \times p$ constant matrices respectively. $x(t)$ is an n-state vector and $u(t)$ is the p-input control vector.

A first order differential algebraic equations of time-varying case is represented in the following form

$$K(t)\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

with initial condition $x(0) = x_0$.

where $x(t)$ and $u(t)$ are defined as above and $K(t)$ is an $n \times n$ singular matrix, $A(t)$ and $B(t)$ are $n \times n$ and $n \times p$ matrices respectively. The elements (not necessarily all the elements) of the matrices $K(t), A(t)$ and $B(t)$ are time dependent.

NUMERICAL EXAMPLES

In this section, the exact solutions and approximated solutions obtained by Rational approximation method and Pade approximation method. To show the efficiency of the Rational approximation method, we have considered the following problem taken from [6], with step size $t = 0.1$ along with the exact solutions.

The discrete solutions obtained by the two methods, Rational approximation method and the Pade approximation method; the absolute errors between them are tabulated and are presented in Table 1 - 2. To distinguish the effect of the errors in accordance with the exact solutions, graphical representations are given for selected values of "t" and are presented in Fig. 1 to Fig. 5 for the following problem, using three dimensional effects.

Example 4.1

The first order differential algebraic equations of time-invariant case with three variables of the form (1) is given by [6]

$$K = \begin{bmatrix} 0 & 1 & 4 \\ 0 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$, A=I (an identify matrix of appropriate dimension) and B = 0

with initial condition $x(0) = [1/6 \quad 1 \quad -1/3]^T$,

and the exact solution is

$$x_1 = (\exp(-t/2))/6$$

$$x_2 = \exp(-t/2)$$

$$x_3 = -(\exp(-t/2))/3$$

Example 4.2

The first order differential algebraic equations of time-varying case with two variables of the form (2) is given by [6]

$$K = \begin{bmatrix} 0 & 0 \\ 1 & t \end{bmatrix}, A = \begin{bmatrix} -1 & 1-t \\ 0 & -2 \end{bmatrix}, B = \begin{bmatrix} e^t & 1 \\ t^2 & 2 \end{bmatrix} \quad \text{and } u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

with initial condition

$$x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and the exact solution is

$$x_1 = (1+t)e^t - t^3$$

$$x_2 = t^2 - e^t$$

using Rational approximation method and Pade approximation method to solve the above problems, the absolute errors are evaluated and are presented in Table 1 and Table 2 with various time step size. Error graphs are presented Fig. 1 to Fig. 6 to highlight the efficiency of the method.

TABLE I

t	Example 4.1					
	Pade approximation Error			Rational approximation Error		
	x_1	x_2	x_3	x_1	x_2	x_3
0.1	1E-09	4E-09	3E-09	1E-11	4E-11	3E-11
0.2	2E-09	6E-09	5E-09	2E-11	6E-11	5E-11
0.3	3E-09	8E-09	7E-09	3E-11	8E-11	7E-11
0.4	4E-09	1E-08	9E-09	4E-11	1E-10	9E-11
0.5	5E-09	1.2E-08	1.1E-08	5E-11	1.2E-10	1.1E-10
0.6	6E-09	1.4E-08	1.3E-08	6E-11	1.4E-10	1.3E-10
0.7	7E-09	1.6E-08	1.5E-08	7E-11	1.6E-10	1.5E-10
0.8	8E-09	1.8E-08	1.7E-08	8E-11	1.8E-10	1.7E-10
0.9	9E-09	2E-08	1.9E-08	9E-11	2E-10	1.9E-10
1.0	1E-08	2.2E-08	2.1E-08	1E-10	2.2E-10	2.1E-10

TABLE III

t	Example 4.2			
	Pade approximation Error		Rational approximation Error	
	x_1	x_2	x_1	x_2
0.1	2E-09	7E-09	2E-11	7E-11
0.2	5E-09	9E-09	5E-11	9E-11
0.3	8E-09	1.1E-08	8E-11	1.1E-10
0.4	1.1E-08	1.3E-08	1.1E-10	1.3E-10
0.5	1.4E-08	1.5E-08	1.4E-10	1.5E-10
0.6	1.7E-08	1.7E-08	1.7E-10	1.7E-10
0.7	2E-08	1.9E-08	2E-10	1.9E-10
0.8	2.3E-08	2.1E-08	2.3E-10	2.1E-10
0.9	2.6E-08	2.3E-08	2.6E-10	2.3E-10
1.0	2.9E-08	2.5E-08	2.9E-10	2.5E-10

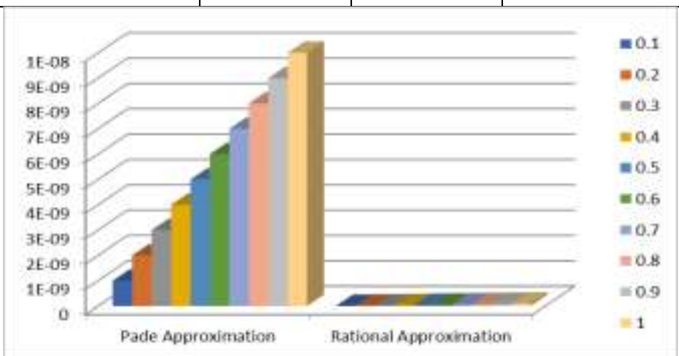

Fig. 1 Error estimation of Example 4.1 at x_1

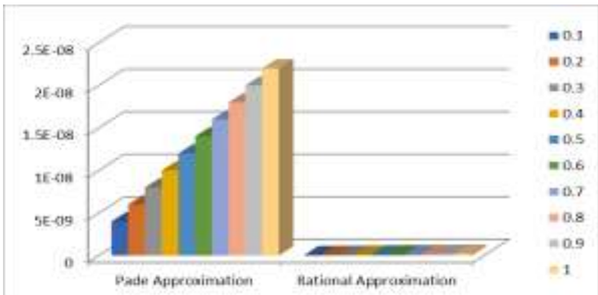

Fig. 2 Error estimation of Example 4.1 at x_2

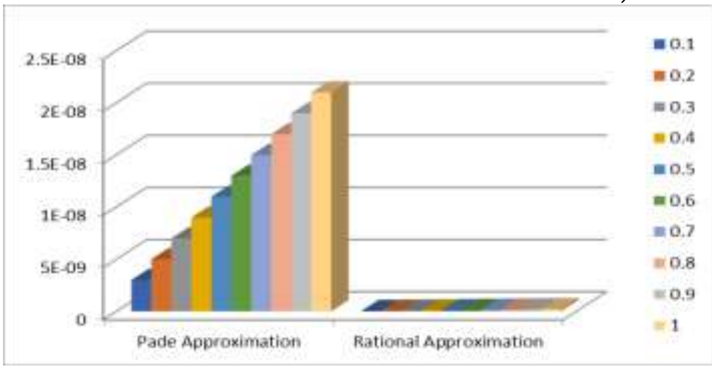

Fig. 3 Error estimation of Example 4.1 at x_3

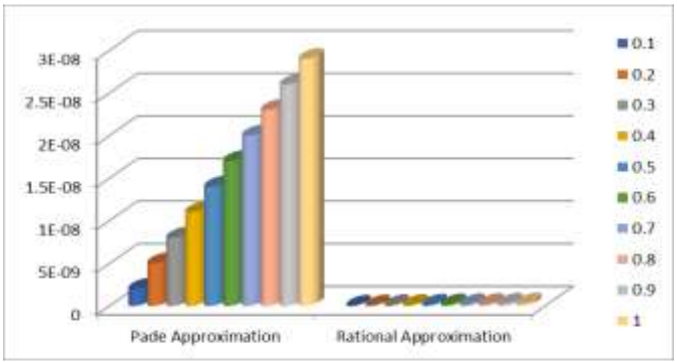

Fig. 4 Error estimation of Example 4.2 at x_1

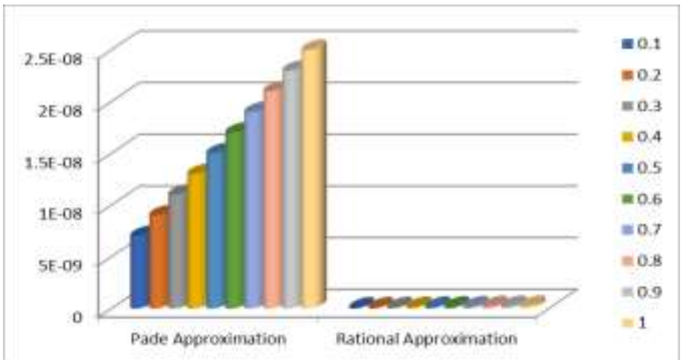

Fig. 6 Error estimation of Example 4.2 at x_2

CONCLUSIONS

A simple and easy method is introduced in this paper to obtain discrete solutions of first order differential algebraic equations of time-invariant and time varying cases using Rational approximation method. The efficiency and the accuracy of the Rational approximation method have been illustrated by suitable examples. The solutions obtained are compared well with the exact solutions and Pade approximation method. It has been observed that the solutions by our method show good agreement with the exact solutions. The present method is very convenient as it requires only simple computing systems, less computing time and less memory. The Rational approximation method is very simple and direct which provides the solutions for any length of time.

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